

SOME EXAMPLES OF ISOMORPHISMS INDUCED BY FOURIER-MUKAI FUNCTORS

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0. INTRODUCTION

In order to investigate sheaves on abelian varieties, Mukai [Mu1] introduced a very powerful tool called Fourier-Mukai functor. As an application, Mukai ([Mu1], [Mu4]) computed some moduli spaces of stable sheaves on abelian varieties. Recently, Dekker [D] found some examples of isomorphisms of moduli spaces of sheaves induced by Fourier-Mukai functors. As an application, he proved that moduli spaces of sheaves on abelian surfaces are deformation equivalent to Hilbert schemes of points (see Theorem 1.4).

Recently Fourier-Mukai functor was generalized to more general situations (e.g. [Br1], [Br2], [Mu6], [Mu7]). Next task is to construct many examples of birational maps of moduli spaces of sheaves. In this note, we restrict ourselves to an abelian or a K3 surface of Picard number 1 and give some examples of birational maps of moduli spaces of sheaves induced by Fourier-Mukai functor (Theorem 2.3). More precisely, we shall find some examples of sheaves which satisfy WIT_i. In general, Fourier-Mukai functor does not induce isomorphisms of moduli spaces of sheaves. Motivated by recent work of Markman [Mr], we also consider the composition of Fourier-Mukai functor and “taking-dual” functor. Then we can get isomorphisms in these cases.

As an application, we get another proof of Dekker’s result (Theorem 1.4). Our condition and method are similar to [Y4]. In section 1, we shall treat original Fourier-Mukai functor. In section 2, we shall explain how to generalize it to more general situations.

Notation. Let X be an abelian or a K3 surface defined over \mathbb{C} . We set $H^{ev}(X, \mathbb{Z}) := \bigoplus_i H^{2i}(X, \mathbb{Z})$. For $x \in H^{ev}(X, \mathbb{Z})$, $[x]_i$ denotes the $2i$ -th component of x . Let $(H^{ev}(X, \mathbb{Z}), \langle \ , \ \rangle)$ be the Mukai lattice of X .

Let $\mathbf{D}(X)$ be the derived category of X . For $x \in \mathbf{D}(X)$,

$$\begin{aligned} v(x) &:= \text{ch}(x) \sqrt{\text{td}_X} \\ &= \text{ch}(x)(1 + \varepsilon \omega) \in H^{ev}(X, \mathbb{Z}) \end{aligned} \quad (0.1)$$

is the Mukai vector of x , where $\varepsilon = 0, 1$ according as X is an abelian surface or a K3 surface, and ω is the fundamental class of X . Let L be an ample line bundle on X . For a Mukai vector v , $M_L(v)$ is the moduli space of stable sheaves E of $v(E) = v$ with respect to L . For a primitive Mukai vector v , $M_L(v)$ is smooth and projective, if L is general (cf. [Y2]).

If $\text{NS}(X) = \mathbb{Z}$, then for a coherent sheaf E on X , we set

$$\deg(E) := \frac{(c_1(E), c_1(L))}{(c_1(L)^2)} \in \mathbb{Z}, \quad (0.2)$$

where L is the ample generator of $\text{NS}(X)$.

1. A SPECIAL CASE

1.1. Preliminaries. We start with original Fourier-Mukai functor. So we assume that X is an abelian surface. Let \widehat{X} be the dual of X and \mathcal{P} the Poincaré line bundle on $X \times \widehat{X}$. We denote the projections $X \times \widehat{X} \rightarrow X$ (resp. $X \times \widehat{X} \rightarrow \widehat{X}$) by p_X (resp. $p_{\widehat{X}}$). Let $\mathcal{F} : \mathbf{D}(X) \rightarrow \mathbf{D}(\widehat{X})$ be the Fourier-Mukai functor defined by

$$\mathcal{F}(x) := \mathbf{R}p_{\widehat{X}*}(\mathcal{P} \otimes p_X^*(x)), x \in \mathbf{D}(X). \quad (1.1)$$

Let $\widehat{\mathcal{F}} : \mathbf{D}(\widehat{X}) \rightarrow \mathbf{D}(X)$ be the inverse of \mathcal{F} :

$$\widehat{\mathcal{F}}(y) := \mathbf{R}p_{X*}(\mathcal{P}^\vee \otimes p_{\widehat{X}}^*(y))[2], y \in \mathbf{D}(\widehat{X}). \quad (1.2)$$

By [Y5, Lem. 4.2], we can define an isometry $\mathcal{F}_H : H^{ev}(X, \mathbb{Z}) \rightarrow H^{ev}(\widehat{X}, \mathbb{Z})$ of Mukai lattices by

$$\mathcal{F}_H(x) := p_{\widehat{X}*}((\text{ch } \mathcal{P}) p_X^* \sqrt{\text{td}_X} p_{\widehat{X}}^* \sqrt{\text{td}_{\widehat{X}}} p_X^*(x)), x \in H^{ev}(X, \mathbb{Z}). \quad (1.3)$$

Then the inverse $\widehat{\mathcal{F}}_H : H^{ev}(\widehat{X}, \mathbb{Z}) \rightarrow H^{ev}(X, \mathbb{Z})$ of \mathcal{F}_H is given by

$$\widehat{\mathcal{F}}_H(y) := p_{X*}((\text{ch } \mathcal{P})^\vee p_X^* \sqrt{\text{td}_X} p_{\widehat{X}}^* \sqrt{\text{td}_{\widehat{X}}} p_X^*(y)), y \in H^{ev}(\widehat{X}, \mathbb{Z}). \quad (1.4)$$

By Grothendieck Riemann-Roch theorem, the following diagram is commutative.

$$\begin{array}{ccc} \mathbf{D}(X) & \xrightarrow{\mathcal{F}} & \mathbf{D}(\widehat{X}) \\ \sqrt{\text{td}_X} \text{ch} \downarrow & & \downarrow \sqrt{\text{td}_{\widehat{X}}} \text{ch} \\ H^{ev}(X, \mathbb{Z}) & \xrightarrow{\mathcal{F}_H} & H^{ev}(\widehat{X}, \mathbb{Z}) \end{array} \quad (1.5)$$

For a coherent sheaf E on X (resp. a coherent sheaf F on \widehat{X}), we set

$$\begin{aligned} \mathcal{F}^i(E) &:= H^i(\mathcal{F}(E)), \\ \widehat{\mathcal{F}}^i(F) &:= H^i(\widehat{\mathcal{F}}(F)). \end{aligned} \quad (1.6)$$

If E (resp. F) satisfies WIT_i with respect to \mathcal{F} (resp. $\widehat{\mathcal{F}}$), then we denote $\mathcal{F}^i(E)$ (resp. $\widehat{\mathcal{F}}^i(F)$) by \widehat{E} (resp. \widehat{F}).

We are also interested in the composition of \mathcal{F} and the “taking-dual” functor $\mathcal{D}_{\widehat{X}} : \mathbf{D}(\widehat{X}) \rightarrow \mathbf{D}(\widehat{X})_{op}$ sending $x \in \mathbf{D}(\widehat{X})$ to $\mathbf{R}\mathcal{H}om(x, \mathcal{O}_{\widehat{X}})$, where $\mathbf{D}(\widehat{X})_{op}$ is the opposite category of $\mathbf{D}(\widehat{X})$. By Grothendieck-Serre duality, $\mathcal{G} := (\mathcal{D}_{\widehat{X}} \circ \mathcal{F})[-2]$ is defined by

$$\mathcal{G}(x) := \mathbf{R}\mathcal{H}om_{p_{\widehat{X}}} (p_X^*(x), \mathcal{P}^\vee), x \in \mathbf{D}(X). \quad (1.7)$$

Let $\widehat{\mathcal{G}} : \mathbf{D}(\widehat{X})_{op} \rightarrow \mathbf{D}(X)$ be the inverse of \mathcal{G} :

$$\widehat{\mathcal{G}}(y) := \mathbf{R}\mathcal{H}om_{p_X} (p_{\widehat{X}}^*(y), \mathcal{P}^\vee), y \in \mathbf{D}(\widehat{X}). \quad (1.8)$$

For a coherent sheaf E on X (resp. a coherent sheaf F on \widehat{X}), we set

$$\begin{aligned} \mathcal{G}^i(E) &:= H^i(\mathcal{G}(E)), \\ \widehat{\mathcal{G}}^i(F) &:= H^i(\widehat{\mathcal{G}}(F)). \end{aligned} \quad (1.9)$$

Then there are spectral sequences

$$E_2^{p,q} = \widehat{\mathcal{G}}^p(\mathcal{G}^{-q}(E)) \Rightarrow \begin{cases} E, & p+q=0 \\ 0, & \text{otherwise,} \end{cases} \quad (1.10)$$

$$E_2^{p,q} = \mathcal{G}^p(\widehat{\mathcal{G}}^{-q}(F)) \Rightarrow \begin{cases} F, & p+q=0 \\ 0, & \text{otherwise.} \end{cases} \quad (1.11)$$

In particular

$$\begin{cases} \mathcal{G}^p(\widehat{\mathcal{G}}^0(F)) = 0, & p = 1, 2, \\ \mathcal{G}^p(\widehat{\mathcal{G}}^2(F)) = 0, & p = 0, 1. \end{cases} \quad (1.12)$$

If E (resp. F) satisfies WIT_i with respect to \mathcal{G} (resp. $\widehat{\mathcal{G}}$), then we denote $\mathcal{G}^i(E)$ (resp. $\widehat{\mathcal{G}}^i(F)$) by \widehat{E} (resp. \widehat{F}).

1.2. The case of $\langle v, 1 \rangle < 0$. We assume that $\text{NS}(X) = \mathbb{Z}L$, where L is an ample generator. Then the dual of X also satisfies the same condition. We set $\widehat{L} := \det(-\mathcal{F}(L))$. Then \widehat{L} is the ample generator of $\text{NS}(\widehat{X})$. For $v = r + dc_1(L) + a\omega \in H^{ev}(X, \mathbb{Z})$, $\mathcal{F}_H(v) = a - dc_1(\widehat{L}) + r\omega$. In this and the next subsections, we shall consider functors \mathcal{F} and \mathcal{G} . In this subsection, we treat the case of $\langle v, 1 \rangle < 0$. We first treat the functor \mathcal{G} .

Proposition 1.1. *Let E be a μ -stable sheaf of Mukai vector $v(E) = r + c_1(L) + a\omega$. If $a > 0$, then E satisfies WIT_2 with respect to \mathcal{G} and \widehat{E} is a μ -stable sheaf of $v(\widehat{E}) = a + c_1(\widehat{L}) + r\omega$. In particular, \mathcal{G} induces an isomorphism $M_L(v) \rightarrow M_{\widehat{L}}(\mathcal{F}_H(v)^\vee)$.*

Proof. (1) E satisfies WIT_2 . By the stability of E , $\mathcal{G}^0(E) = 0$. Hence we shall prove that $\mathcal{G}^1(E) = 0$. We first show that $\text{Ext}^1(E, \mathcal{P}_x^\vee) = 0$ except for finitely many points $x \in \widehat{X}$. Suppose that $\text{Ext}^1(E, \mathcal{P}_x^\vee) \neq 0$ for distinct points x_1, x_2, \dots, x_n . By [Y4, Lem. 2.1], we get a μ -stable extension sheaf G :

$$0 \rightarrow \bigoplus_{i=1}^n \mathcal{P}_{x_i}^\vee \rightarrow G \rightarrow E \rightarrow 0. \quad (1.13)$$

Since $v(G) = v(E) + n$, we see that $\langle v(G)^2 \rangle = \langle v(E)^2 \rangle - 2na$. Hence n must satisfy the inequality $n \leq \langle v(E)^2 \rangle / 2a$.

By using this, we prove that $\mathcal{G}^1(E) = 0$. By base change theorem, $\mathcal{G}^1(E)$ is of dimension 0. Hence we show that $\widehat{\mathcal{G}}^2(\mathcal{G}^1(E)) = 0$. Since $\mathcal{G}^0(E) = 0$, $\widehat{\mathcal{G}}^0(\mathcal{G}^0(E)) = 0$. By using the spectral sequence (1.10), we conclude that $\widehat{\mathcal{G}}^2(\mathcal{G}^1(E)) = 0$.

(2) \widehat{G} is torsion free.¹ Indeed, let T be the torsion submodule of \widehat{E} . Since \widehat{E} is locally free in codimension 1, T is of dimension 0. Hence T satisfies IT_2 and $\deg(\widehat{\mathcal{G}}^2(T)) = 0$. Since $\widehat{\mathcal{G}}^2(T)$ is a quotient of E , we get a contradiction.

(3) \widehat{E} is μ -stable. If \widehat{E} is not μ -stable, then there is an exact sequence

$$0 \rightarrow A \rightarrow \widehat{E} \rightarrow B \rightarrow 0, \quad (1.14)$$

where B is a μ -stable sheaf of $\deg(B) \leq 0$. Then we get

$$\widehat{\mathcal{G}}^0(B) = 0, \quad (1.15)$$

$$\widehat{\mathcal{G}}^1(B) = \widehat{\mathcal{G}}^0(A), \quad (1.16)$$

$$(1.17)$$

and an exact sequence

$$0 \rightarrow \widehat{\mathcal{G}}^1(A) \rightarrow \widehat{\mathcal{G}}^2(B) \rightarrow E \rightarrow \widehat{\mathcal{G}}^2(A) \rightarrow 0. \quad (1.18)$$

If $B \neq \mathcal{P}_x^\vee$ for any $x \in X$, then $\widehat{\mathcal{G}}^2(B) = 0$. Hence B satisfies WIT_1 . By (1.12), $\mathcal{G}^1(\widehat{\mathcal{G}}^2(B)) = \mathcal{G}^1(\widehat{\mathcal{G}}^0(A)) = 0$. Hence $B = 0$, which is a contradiction. If $B = \mathcal{P}_x^\vee$ for some $x \in X$, then $\widehat{\mathcal{G}}^1(B) = 0$ and $\widehat{\mathcal{G}}^2(B) \cong \mathbb{C}_x$. Hence $\widehat{\mathcal{G}}^0(A) = 0$ and $\widehat{\mathcal{G}}^1(A) = \widehat{\mathcal{G}}^2(B) = \mathbb{C}_x$. So we get $\mathcal{G}^2(\widehat{\mathcal{G}}^1(A)) \neq 0$, which contradicts (1.11). \square

By the proof of this proposition, E satisfies IT_2 for \mathcal{G} if $a > \langle v^2 \rangle / 2$. Hence E also satisfies IT_0 for \mathcal{F} . Since $\mathcal{F}^0(E) \cong \mathcal{G}^2(E)^\vee$, $\mathcal{F}^0(E)$ is also μ -stable. Thus we get the following.

Corollary 1.2. *Let E be a μ -stable sheaf of Mukai vector $v(E) = r + c_1(L) + a\omega$. We assume that $a > \langle v^2 \rangle / 2$. Then E satisfies IT_0 with respect to \mathcal{F} and \widehat{E} is a μ -stable vector bundle of $v(\widehat{E}) = a - c_1(\widehat{L}) + r\omega$.*

Since $\widehat{\mathcal{F}} = \widehat{\mathcal{G}} \circ \mathcal{D}_{\widehat{X}}$, we also obtain the following.

Corollary 1.3. *Let E be a μ -stable vector bundle of Mukai vector $v(E) = r - c_1(\widehat{L}) + a\omega$ on \widehat{X} . We assume that $a > 0$. Then E satisfies WIT_2 with respect to $\widehat{\mathcal{F}}$ and \widehat{E} is μ -stable.*

In the same way as in the proof of [Y4, Thm. 3.6], we get another proof of Dekker's result [D, Thm. 5.8].

Theorem 1.4. *Let X be an arbitrary abelian surface. Let $v = r + \xi + a\omega, \xi \in H^2(X, \mathbb{Z})$ be a Mukai vector such that $r + \xi$ is primitive. Then $M_L(v)$ is deformation equivalent to $\widehat{X} \times \text{Hilb}_X^{\langle v^2 \rangle / 2}$ for a general ample divisor L .*

1.3. The case of $\langle v, 1 \rangle > 0$. As in the previous subsection, we assume that $\text{NS}(X) = \mathbb{Z}L$.

Proposition 1.5. *Let E be a μ -stable sheaf of Mukai vector $v(E) = r + c_1(L) + a\omega$. We assume that $a < 0$. Then E satisfies WIT_1 and \widehat{E} is a μ -stable sheaf of $v(\widehat{E}) = -a + c_1(\widehat{L}) - r\omega$. In particular, Fourier-Mukai functor induces an isomorphism $M_L(v) \rightarrow M_{\widehat{L}}(-\mathcal{F}_H(v))$.*

Proof. (1) E satisfies WIT_1 . We first show that $H^0(X, E \otimes \mathcal{P}_x) = 0$ except for finitely many points $x \in \widehat{X}$. Suppose that $k_i := h^0(X, E \otimes \mathcal{P}_{x_i}) \neq 0$ for distinct points x_1, x_2, \dots, x_n . We shall consider the evaluation map

$$\phi : \bigoplus_{i=1}^n \mathcal{P}_{x_i}^\vee \otimes H^0(X, E \otimes \mathcal{P}_{x_i}) \rightarrow E. \quad (1.19)$$

¹ This claim also follows from the proof of base change theorem.

We assume that $\sum_i k_i > r$, that is, $\text{rk}(\oplus_{i=1}^n \mathcal{P}_{x_i}^\vee \otimes H^0(X, E \otimes \mathcal{P}_{x_i})) > \text{rk}(E)$. By the proof of [Y4, Lem. 2.1], ϕ is surjective in codimension 1 and $\ker \phi$ is μ -stable. We set $b := \dim(\text{coker } \phi)$. Then $v(\ker \phi) = \sum_{i=1}^n k_i - (v(E) - b\omega)$. Since $\sum_i k_i > r$, we get

$$\begin{aligned} \langle v(\ker \phi)^2 \rangle &= \langle v(E)^2 \rangle + 2a \sum_i k_i - 2b \sum_i k_i + 2br \\ &\leq \langle v(E)^2 \rangle + 2a \sum_i k_i. \end{aligned} \quad (1.20)$$

Since $\langle v(\ker \phi)^2 \rangle \geq 0$, we get $\sum_i k_i \leq \langle v(E)^2 \rangle / (-2a)$. In particular, $\mathcal{F}^0(E)$ is a torsion sheaf. Since E is a torsion free sheaf on the integral scheme $\text{Supp}(E)$, $\mathcal{F}^0(E)$ is torsion free. Hence we get $\mathcal{F}^0(E) = 0$. By the stability of E , $\mathcal{F}^2(E) = 0$. Therefore E satisfies WIT_1 .

(2) \hat{E} is torsion free. Let T be the torsion subsheaf of \hat{E} . By (1), T is of dimension 0. Since \hat{E} satisfies WIT_1 and T satisfies IT_0 , T must be 0.

(3) \hat{E} is μ -stable. Assume that \hat{E} is not μ -stable. Let $0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = \hat{E}$ be the Harder-Narasimhan filtration of \hat{E} . We shall choose the integer k which satisfies $\deg(F_i/F_{i-1}) > 0, i \leq k$ and $\deg(F_i/F_{i-1}) \leq 0, i > k$. We shall prove that $\mathcal{F}^2(F_k) = 0$ and $\mathcal{F}^0(\hat{E}/F_k) = 0$. Since $\deg(F_i/F_{i-1}) > 0, i \leq k$, semi-stability of F_i/F_{i-1} implies that $\mathcal{F}^2(F_i/F_{i-1}) = 0, i \leq k$. Hence $\mathcal{F}^2(F_k) = 0$. On the other hand, we also see that $\mathcal{F}^0(F_i/F_{i-1}), i > k$ is of dimension 0. Since F_i/F_{i-1} is torsion free, $\mathcal{F}^0(F_i/F_{i-1}) = 0, i > k$. Hence we conclude that $\mathcal{F}^0(\hat{E}/F_k) = 0$.

So F_k and \hat{E}/F_k satisfy WIT_1 and we get an exact sequence

$$0 \rightarrow \mathcal{F}^1(F_k) \rightarrow E \rightarrow \mathcal{F}^1(\hat{E}/F_k) \rightarrow 0. \quad (1.21)$$

Since $\deg(\mathcal{F}^1(F_k)) = \deg(F_k) > 0$, μ -stability of E implies that $\deg(\mathcal{F}^1(F_k)) = 1$ and $\text{rk}(\mathcal{F}^1(F_k)) = \text{rk}(E)$. Thus $\mathcal{F}^1(\hat{E}/F_k)$ is of dimension 0. Then $\mathcal{F}^1(\hat{E}/F_k)$ satisfies IT_0 , which is a contradiction. \square

2. MORE GENERAL CASES

In this section, we treat more general cases. Let (X, L) be a polarized abelian (or K3) surface of $(L^2) = 2r_0k$, where r_0 and k are positive integers of $(r_0, k) = 1$. We assume that $\text{NS}(X) = \mathbb{Z}L$.² We set $v_0 := r_0 + d_0c_1(L) + d_0^2k\omega$, where d_0 is an integer of $(r_0, d_0) = 1$. Then $\langle v_0^2 \rangle = 0$. So $Y := M_L(v_0)$ is an abelian (or K3) surface. Since X and Y are isogenous, $\text{NS}(Y) \cong \mathbb{Z}$. Since $(r_0, d_0^2k) = 1$, there is a universal family \mathcal{E} on $X \times Y$. We assume that

- \mathcal{E} is locally free.³

We set

$$\hat{L} := \det(p_{Y!}(\mathcal{E} \otimes \mathcal{O}_L(kr_0 - 2kd_0)))^\vee. \quad (2.1)$$

Then \hat{L} is a primitive ample line bundle of $(c_1(\hat{L})^2) = (c_1(L)^2)$. Indeed, v_0^\vee and $v(\mathcal{O}_L(kr_0 - 2kd_0)) = c_1(L) - 2kd_0\omega$ generate $(v_0^\vee)^\perp$. Since

$$\theta_{v_0^\vee} : (v_0^\vee)^\perp / \mathbb{Z}v_0^\vee \rightarrow H^2(Y, \mathbb{Z}) \quad (2.2)$$

is an isomorphism, $c_1(\hat{L})$ is primitive. By Donaldson, \hat{L} is ample. Hence \hat{L} is a primitive ample line bundle. Since $\mathcal{F}^{v_0}(\mathcal{O}_L(kr_0 - 2kd_0)) = c_1(\hat{L}) + b\hat{\omega}, b \in \mathbb{Z}$ and $\mathcal{F}_H^{v_0}$ is an isometry of Mukai lattice, $(c_1(\hat{L})^2) = \langle v(\mathcal{F}^{v_0}(\mathcal{O}_L(kr_0 - 2kd_0)))^2 \rangle = \langle v(\mathcal{O}_L(kr_0 - 2kd_0))^2 \rangle = (c_1(L)^2)$.

Let $\mathcal{F}^{v_0} : \mathbf{D}(X) \rightarrow \mathbf{D}(Y)$ be the Fourier-Mukai functor defined by \mathcal{E} and $\mathcal{F}_H^{v_0} : H^{ev}(X, \mathbb{Z}) \rightarrow H^{ev}(Y, \mathbb{Z})$ the induced isometry.

Lemma 2.1. *Let d_1 and l be integers which satisfy $d_1(kd_0) - lr_0 = 1$. Then replacing \mathcal{E} by $\mathcal{E} \otimes p_Y^*N$, $N \in \text{Pic}(Y)$, we get*

$$\begin{cases} \mathcal{F}_H^{v_0}(1) = d_0^2k + d_0lc_1(\hat{L}) + l^2r_0\hat{\omega} \\ \mathcal{F}_H^{v_0}(c_1(L)) = 2d_0kr_0 + (2d_0kd_1 - 1)c_1(\hat{L}) + (2d_0k^2d_1^2 - 2d_1k)\hat{\omega} \\ \mathcal{F}_H^{v_0}(\omega) = r_0 + d_1c_1(\hat{L}) + d_1^2k\hat{\omega}, \end{cases} \quad (2.3)$$

²Under this assumption, every simple vector bundle of isotropic primitive Mukai vector is stable (cf. [Mu3]).

³ If \mathcal{E} is not locally free, then Fourier-Mukai functor is the same as reflection functor [Mu3]. This case was treated in [Mr] and [Y4].

where $\widehat{\omega}$ is the fundamental class of Y .

Proof. We set

$$\begin{cases} [\mathcal{F}_H^{v_0}(1)]_1 = ac_1(\hat{L}), \\ [\mathcal{F}_H^{v_0}(c_1(L))]_1 = bc_1(\hat{L}), \\ [\mathcal{F}_H^{v_0}(\omega)]_1 = cc_1(\hat{L}). \end{cases} \quad (2.4)$$

It is easy to see that $[\mathcal{F}_H^{v_0}(v_0^\vee)]_1 = 0$. Hence we get the relation

$$r_0a - d_0b + d_0^2kc = 0. \quad (2.5)$$

Since $(r_0, d_0) = 1$, $b \equiv d_0kc \pmod{r_0}$. By the definition of \hat{L} , $-b + 2kd_0c = 1$. Hence we get $kd_0c \equiv 1 \pmod{r_0}$. So replacing \mathcal{E} by $\mathcal{E} \otimes \hat{L}^{\otimes((d_1-c)/r_0)}$, we may assume that

$$\begin{cases} [\mathcal{F}_H^{v_0}(c_1(L))]_1 = (2kd_0d_1 - 1)c_1(\hat{L}), \\ [\mathcal{F}_H^{v_0}(\omega)]_1 = d_1c_1(\hat{L}). \end{cases} \quad (2.6)$$

Since $\mathcal{F}_H^{v_0}$ is an isometry, $\langle \mathcal{F}_H^{v_0}(\omega)^2 \rangle = \langle \omega^2 \rangle = 0$. Hence we get

$$\mathcal{F}_H^{v_0}(\omega) = r_0 + d_1c_1(\hat{L}) + d_1^2k\widehat{\omega}. \quad (2.7)$$

Since \mathcal{E} is a universal family of stable sheaves of Mukai vector v_0 , we get the following relations:

$$\begin{cases} \mathcal{F}_H^{v_0}(v_0^\vee) = \widehat{\omega} \\ \mathcal{F}_H^{v_0}(\omega) = r_0 + d_1c_1(\hat{L}) + d_1^2k\widehat{\omega} \\ \mathcal{F}_H^{v_0}(-c_1(L) + 2kd_0\omega) = x + c_1(\hat{L}) + y\widehat{\omega}, \end{cases} \quad (2.8)$$

where $x, y \in \mathbb{Z}$. Since $\mathcal{F}_H^{v_0}$ is an isometry, we see that $x = 0$ and $y = -2kd_1$. Hence we get our lemma. \square

In order to generalize Proposition 1.1, and 1.5, let us introduce some notations. Let G be a locally free sheaf on X . For a torsion free sheaf E on X , we define

$$\begin{aligned} \text{rk}_G(E) &:= \text{rk}(E \otimes G^\vee), \\ \deg_G(E) &:= \deg(E \otimes G^\vee), \\ \mu_G(E) &:= \frac{\deg_G(E)}{\text{rk}_G(E)}. \end{aligned} \quad (2.9)$$

For $x \in \mathbf{D}(X)$ (resp. $v(x) \in H^{ev}(X, \mathbb{Z})$), we can also define $\text{rk}_G(x)$ and $\deg_G(x)$ (resp. $\text{rk}_G(v(x))$ and $\deg_G(v(x))$).

Then we see that

$$\begin{aligned} \mu_G(E) &= \frac{\deg(E) \text{rk}(G) - \deg(G) \text{rk}(E)}{\text{rk}(G) \text{rk}(E)} \\ &= \mu(E) - \mu(G). \end{aligned} \quad (2.10)$$

Hence E is μ -stable if and only if

$$\mu_G(F) < \mu_G(E) \quad (2.11)$$

for any subsheaf $F \subset E$ of $\text{rk}(F) < \text{rk}(E)$. Assume that $\deg_G(E) = 1$. Then it is easy to see that E is μ -stable if and only if $\deg_G(F) \leq 0$ for any subsheaf $F \subset E$ of $\text{rk}(F) < \text{rk}(E)$.

Lemma 2.2. *We choose points $s \in X$ and $t \in Y$. We set*

$$\begin{aligned} G_1 &:= \mathcal{E}_{|X \times \{t\}}^\vee, \\ G_2 &:= \mathcal{E}_{|\{s\} \times Y}. \end{aligned} \quad (2.12)$$

Then for a Mukai vector v ,

$$\deg_{G_1}(v) = -\deg_{G_2}(\mathcal{F}_H^{v_0}(v)) = \deg_{G_2^\vee}(\mathcal{F}_H^{v_0}(v)^\vee). \quad (2.13)$$

Proof. We set

$$\begin{aligned} v &= r + dc_1(L) + a\omega, \\ \mathcal{F}_H^{v_0}(v) &= r' + d'c_1(L) + a'\omega. \end{aligned} \quad (2.14)$$

It is sufficient to prove that $r'd_1 - d'r_0 = dr_0 + rd_0$. This follows from the following relations which comes from Lemma 2.1:

$$\begin{cases} r' = r(d_0^2 k) + d(2d_0 r_0 k) + ar_0, \\ d' = r(d_0 l) + d(2d_0 d_1 k - 1) + ad_1. \end{cases} \quad (2.15)$$

□

Due to this lemma, we can use the same arguments as in Propositions 1.1 and 1.5. Hence we get the following theorem.

Theorem 2.3. *Keep the notations as above. Let $v := r + dc_1(L) + a\omega$ be a Mukai vector of $dr_0 + rd_0 = 1$.*

1. *If $-\langle v, v_0^\vee \rangle > 0$, then the composition of Fourier-Mukai functor and “taking-dual” functor \mathcal{D}_Y induces an isomorphism $M_L(v) \rightarrow M_{\hat{L}}(\mathcal{F}_H^{v_0}(v)^\vee)$.*
2. *If $\langle v, v_0^\vee \rangle > 0$, then Fourier-Mukai functor induces an isomorphism $M_L(v) \rightarrow M_{\hat{L}}(-\mathcal{F}_H^{v_0}(v))$.*

Example 1. Keep the notations as above. We set

$$\begin{cases} d_0 = -(r_0 - 1), \\ r = d = 1, \\ k = -n + sr_0, \\ a = (r_0^2 - 1)s - r_0 n, \end{cases} \quad (2.16)$$

where $s > 0$, $sr_0 > n > 0$ and $(r_0, n) = 1$. Then $\langle v^2 \rangle = 2s$ and $\langle v, v_0^\vee \rangle = n > 0$. Applying Theorem 2.3, we get an isomorphism $M_L(v) \rightarrow M_{\hat{L}}(-\mathcal{F}_H^{v_0}(v))$. In particular, we get another proof of [Y4, Thm. 0.2] and Theorem 1.4.

Example 2. We assume that X is a K3 surface and $(L^2) = 12$. We set $v_0 := 2 - L + 3\omega$ and $Y := M_L(v_0)$. Then Y is a K3 surface of $H^2(Y, \mathbb{Z}) \cong v_0^\perp / \mathbb{Z}v_0$. In general, $Y \neq X$. By Fourier-Mukai functor defined by v_0 , we get an isomorphism $M_L(1 + L + 3\omega) \cong M_{\hat{L}}(3 - \hat{L} + \hat{\omega})$. By reflection $R_{v(\mathcal{O}_Y)}$ ([Mr], [Y4]), we obtain a birational map $M_{\hat{L}}(3 - \hat{L} + \hat{\omega}) \cdots \rightarrow M_{\hat{L}}(1 + \hat{L} + 3\hat{\omega})$. Hence we get a birational map $\text{Hilb}_X^4 \cdots \rightarrow \text{Hilb}_Y^4$. Computing ample cones, we see that this birational map is the elementary transformation along \mathbb{P}^2 -bundle over $X \times Y$. In this case, by Torelli Theorem for K3 surfaces, we see that $\text{Hilb}_X^4 \not\cong \text{Hilb}_Y^4$.

3. APPENDIX

Lemma 3.1. *Keep the notation in section 2. Let E be a μ -stable locally free sheaf of $\deg_{G_1}(E) = 0$ and $E \notin M_L(v_0^\perp)$. Then E satisfies IT₁ and \hat{E} is a μ -stable locally free sheaf.*

Proof. Since $E \notin M_L(v_0^\perp)$ and E is μ -stable, we see that E satisfies IT₁. In the same way as in the proof of Lemma 1.5, we see that \hat{E} is μ -semi-stable. Assume that there is an exact sequence

$$0 \rightarrow F_1 \rightarrow \hat{E} \rightarrow F_2 \rightarrow 0, \quad (3.1)$$

where F_2 is a μ -stable sheaf of $\deg_{G_2}(F_2) = 0$ and $\text{rk}(F_2) < \text{rk}(\hat{E})$. Then F_2 satisfies IT₁ and we get an exact sequence

$$0 \rightarrow \mathcal{F}^1(F_1) \rightarrow E \rightarrow \mathcal{F}^1(F_2) \rightarrow \mathcal{F}^2(F_1) \rightarrow 0. \quad (3.2)$$

Since $\mathcal{F}^2(F_1)$ is of dimension 0, $\deg_{G_1}(\mathcal{F}^1(F_1)) = 0$. By the μ -stability of E , we get $\mathcal{F}^1(F_1) = 0$ or $\text{rk}(\mathcal{F}^1(F_1)) = \text{rk}(E)$. We first assume that $\mathcal{F}^1(F_1) = 0$. Since E and $\mathcal{F}^1(F_2)$ are locally free, $\mathcal{F}^2(F_1) = 0$. Hence $F_1 = 0$, which is a contradiction. We next assume that $\text{rk}(\mathcal{F}^1(F_1)) = \text{rk}(E)$. Since $\mathcal{F}^1(F_2)$ is locally free, $\mathcal{F}^1(F_1) \rightarrow E$ is an isomorphism. Hence $\mathcal{F}^1(F_2) \cong \mathcal{F}^2(F_1)$. Since $\mathcal{F}^2(F_1)$ is of dimension 0, we obtain $\mathcal{F}^1(F_2) = 0$. Therefore $F_2 = 0$, which is a contradiction. □

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